Section 5.6 Convolution

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System transfer function

$$\phi(s) = \frac{Y(s)}{F(s)}$$

$f(t)$ known input

$y(t)$ known output

$\hat{f}(t)$ New input, what is the new output $\hat{y}(t)$?

$$\hat{Y}(s) = \phi(s)\,\hat{F}(s)$$

$$y(t) = \mathcal{L}^{-1}\{\hat{Y}(s)\} = \mathcal{L}^{-1}\{\phi(s)\,\hat{F}(s)\}$$

Convolution integral

Two function $f(t)$ & $g(t)$ with Laplace transforms $F(s)$ & $G(s)$

convolution

$$\underbrace{\mathcal{L}^{-1}\{F(s)\,G(s)\}} = \underbrace{(f * g)(t)}$$

$$= \int_0^t f(t-\lambda)\,g(\lambda)\,d\lambda$$

Ex: $\underbrace{\overset{f}{e^{3t}} * \overset{g}{e^{-t}}} = \int_0^t e^{3(t-\lambda)} e^{-\lambda}\,d\lambda$

$$= \int_0^t e^{3t-3\lambda} e^{-\lambda}\,d\lambda$$

$u = 3t - 4\lambda$ $\quad = -\frac{1}{4}\int^{\lambda=t} e^{3t-4\lambda}(-4\,d\lambda)$

$u = 3t - 4\lambda$

$du = -4\,d\lambda$

$$= -\frac{1}{4}\int_{\lambda=0}^{\lambda=t} e^{3t-4\lambda}\,(-4\,d\lambda)$$

$$= -\frac{1}{4}\left[e^{3t-4\lambda}\right]_{\lambda=0}^{\lambda=t}$$

$$\boxed{= -\frac{1}{4}\left[e^{-t} - e^{3t}\right]}$$

OR $e^{3t} * e^{-t} = \mathcal{L}^{-1}\left\{\mathcal{L}\{e^{3t}\}\,\mathcal{L}\{e^{-t}\}\right\}$

$$\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha}$$

$$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3} \qquad \mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$$

$$e^{3t} * e^{-t} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{(s-3)} - \frac{1}{4}\frac{1}{(s+1)}\right\}$$

$$\frac{1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} \qquad \boxed{= \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}}$$

$1 = A(s+1) + B(s-3)$

$s = -1 \qquad 1 = -4B$

$B = -\frac{1}{4}$

$s = 3 \qquad 1 = 4A$

$A = \frac{1}{4}$

Ex: $t * \cos(5t) = \int^{t} (t-\lambda)\cos(5\lambda)\,d\lambda$

Ex: $t * \cos(5t) = \int_0^t (t-\lambda)\cos(5\lambda)\,d\lambda$

$u = t-\lambda \quad dv = \cos(5\lambda)\,d\lambda$

$du = -d\lambda \quad v = \frac{1}{5}\sin(5\lambda)$

$= \frac{1}{5}(t-\lambda)\sin(5\lambda) + \frac{1}{25}\int_0^t \sin(5\lambda)(5\,d\lambda)$

$= \frac{1}{5}(t-\lambda)\sin 5\lambda - \frac{1}{25}\cos(5\lambda)\Big|_0^t$

$= \frac{1}{5}\cancel{(t-t)}^{0}\sin 5t - \frac{1}{25}\cos(5t) - \left[\frac{1}{5}\,t\,\cancel{\sin}^{0}(0) - \frac{1}{25}\cancel{\cos}^{1}(0)\right]$

$$\boxed{= -\frac{1}{25}\cos(5t) + \frac{1}{25}}$$

$$= \left\{\frac{1}{s^2}\cdot\frac{5}{s^2+25}\right\}$$

A spring/mass/dashpot system has mass 1 kg, damping constant 8 kg/sec and spring constant 12 kg per sq sec. The system starts at rest and then has an external force of e^{-5t} Newtons applied after t seconds. The IVP below models the system:

$x'' + 8x' + 12x = e^{-5t}, \quad x(0) = 0, \quad x'(0) = 0$

The Laplace transform of the IVP has solution $Y(s) = F(s)\cdot\Phi(s)$ where $F(s)$ represents the Laplace transform of the forcing term e^{-5t} and $\Phi(s)$ represents the transfer function.

$\Phi(s) =$ [ ] Preview

The weight function $w(t)$ is the inverse Laplace transform of the transfer function.

$w(t) = L^{-1}(\Phi(s)) =$ [ ] Preview

The solution to the IVP is the convolution of the forcing term with the weight function.

$f(t)*w(t) =$ [ ] Preview

$\mathcal{L}\{x'' + 8x' + 12x\} = \mathcal{L}\{e^{-5t}\}$

$s^2X(s) + 8sX(s) + 12X(s) = \frac{1}{s+5}$

$X(s) = \underbrace{\left(\frac{1}{s^2+8s+12}\right)}\left(\frac{1}{s+5}\right) = \Phi(s)F(s)$

$$\mathcal{L}^{-1}\{\Phi(s)\} = \mathcal{L}\left\{\frac{1}{(s+6)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s+6}+\frac{B}{s+2}\right\}$$

$$w(t) = A e^{-6t} + B e^{-2t}$$ $\leftarrow$ Find A&B

$$y(t) = f(t) * w(t) = \int_0^t e^{-5(t-\lambda)}\left[A e^{-6\lambda} + B e^{-2\lambda}\right] d\lambda$$

$$= \int_0^t A e^{-5(t-\lambda)} e^{-6\lambda}\, d\lambda + \int_0^t B e^{-5(t-\lambda)} e^{-2\lambda}\, d\lambda$$